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MINIMIZATION ON STOCHASTIC MATROIDS

Michael P. Bailey

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MINIMIZATION ON STOCHASTIC MATROIDS

by:
Michael P. Bailey
Naval Postgraduate School
Monterey, CA 93943-5000

Abstract

This work gives a methodology for analyzing matroids with random element weights, with emphasis placed on independent, exponentially distributed element weights. The minimum weight basic element in such a structure is shown to be an absorbing state in a Markov chain, while the distribution of weight of the minimum weight element is shown to be of phase-type. We then present two sided bounds for matriods with NBUE distributed weights, as well as for weights with bounded positive hazard rates. We illustrate our method using the transversal matroid to solve stochastic assignment problems.

KEY WORDS: stochastic matriod, stochastic spanning tree, NBUE distributions

1. INTRODUCTION

Matroid theory has its origins in the recognition by Whitney [1935] that some algebraic systems shared several properties with linear independence systems. These systems, called *matroid independence systems*, were shown by Edmonds [1971] to have a close relationship to the greedy algorithm for linear objective functions. The minimum weight spanning tree, the maximum weight transversal, and the fractional knapsack problem are three of the well known combinatorial optimization problems with matroid structures. This paper treats stochastic versions of these problems where element weights are independent, exponentially distributed random variables. We also establish bounds on the expected objective function value of the optimal basic element for new-better-than-used (NBUE) weights, and for weights with positive bounded hazard rates.

There exists some literature on the subject of randomly weighted minimum weight spanning trees, where the weights are i.i.d. exponentials. In Freize [1985], the minimum weight spanning tree on complete graphs with i.i.d. arc weights was considered. It was shown that the weight of the spanning tree approaches a constant as the number of nodes increases.

Mamer and Jain [1988] provided bounding arguments for spanning trees in networks using the so-called *exodic* spanning tree. Corea [1989], and Weiss [1986] indicate how bounding can be done on shortest path systems using phase-type distributions. We develop bounds for matriod minimizations with NBUE weights, where the required phase-type distribution is the one we develop in this paper.

Kulkarni [1988] gave a method for finding the distribution of the weight of the minimum spanning tree in an arbitrary network with independent, exponentially

distributed weights. He provided methods for finding a variety of measures of performance of the spanning tree based on properties of a constructed Markov process. The methodology in the current paper extends the work done by Kulkarni to general matroid structures.

2. COMBINATORIAL UNDERPINNINGS

In this section, we briefly discuss some combinatorial properties of matroids given in standard references such as Lawler [1976]. We then give several results which are essential to our discussion of matroids with random weights.

Let E be a finite set of objects such as vectors, nodes, or arcs. $\mathcal{M}$ is a set of subsets of E with the following two properties:

2.1) $Y \in \mathcal{M}$ and $X \subset Y$, then $X \in \mathcal{M}$

2.2) $\{X \subset A\text{: } X \in \mathcal{M}\text{: there exists no } x \in A \text{ such that } X \cup \{x\} \in \mathcal{M}\}$

is an equicardinal set for every subset A of E.

Property 2.1 says every subset of a set in $\mathcal{M}$ is in $\mathcal{M}$, thus $\mathcal{M}$ is called *simplical*. Property 2.2 dictates that every maximal feasible subset of a set A contains the same number of elements for every $A \subset E$. We will denote the set of maximal elements in $\mathcal{M}$ as $\beta_{\mathcal{M}}$, called the *basis of* $\mathcal{M}$ and will call members of $\beta_{\mathcal{M}}$ *basic* elements. We will consistently use n to denote the cardinality of a basic element.

Properties 2.1 and 2.2 combine to guarantee that we can begin with the empty set ϕ, and construct any set in $\beta_{\mathcal{M}}$ by making n selections from the set E. We will perform this construction of a basic element by greedy minimization.

Let v be a nonnegative weight function on the set E, $v: E \rightarrow \Re^+$. The linear objective function ω on elements of $\mathcal{M}$ is given by

$$\omega(x) = \sum_{x \in X} v(x)$$

for each $X \in \mathcal{M}$. For the time being, we will ignore the possibility of sets of equal weights. In the sequel, the weight of each element of E will be an absolutely continuous random variable, so that equally weighted strings occurr with probability zero. The notion of greediness is formalized by the following algorithm:

0. initialize: $X = \phi$, $\omega = 0$
1. $x \leftarrow \arg\min_{y:\, X \cup y \in \mathcal{M}} v(y)$
2. $w \leftarrow w + v(x)$
3. $X \leftarrow X \cup x$
4. if $X \notin \beta_{\mathcal{M}}$ then go to step 1
5. stop

Figure 2.1. The Greedy Algorithm

Verbally, the greedy algorithm begins with the empty set $X = \phi$ and at each stage selects the element $x \in E - X$ with smallest weight, subject to the constraint that $X \cup \{x\}$ is in $\mathcal{M}$. Let X^G be the basic element constructed by the greedy algorithm, $X^G = \{x_1^G, x_2^G, \ldots, x_n^G\}$ where x_i^G is the element selected at the i^{th} opportunity. Let $X_i^G = \{x_1^G, x_2^G, \ldots, x_i^G\}$ be the set of the first i greedy selections. Note that the terminal value of ω is equal to $\omega(X^G)$, the linear objective function value of X^G. The critical connection between the greedy algorithm and matroid structures is given by the following theorem, directly adapted from results of Edmonds.

Theorem 2.1. Let ω be a linear objective function for an arbitrary nonnegative weight function v, then $X^G = \arg\min_{Y \in \beta_{\mathcal{M}}} \omega(Y)$ if and only if $\mathcal{M}$ is a matroid.

Proof: See Lawler Section 8.3, replacing maximization with minimization throughout. •

As will become apparent, we will need to modify the objective function and greedy algorithm in order to handle stochastic element weights. Let us define an alternate objective function ω_d, which we will call the **discounted** linear objective function, as follows:

$$\omega_d(X) = \sum_{x \in X} v(x) + (n - |X|)\left[\max_{x \in X} v(x)\right].$$

Consider Figure 2.2, with v(x) indicated by the lengths of the bars shown, for $x_1^G, x_2^G, \ldots, x_n^G$.

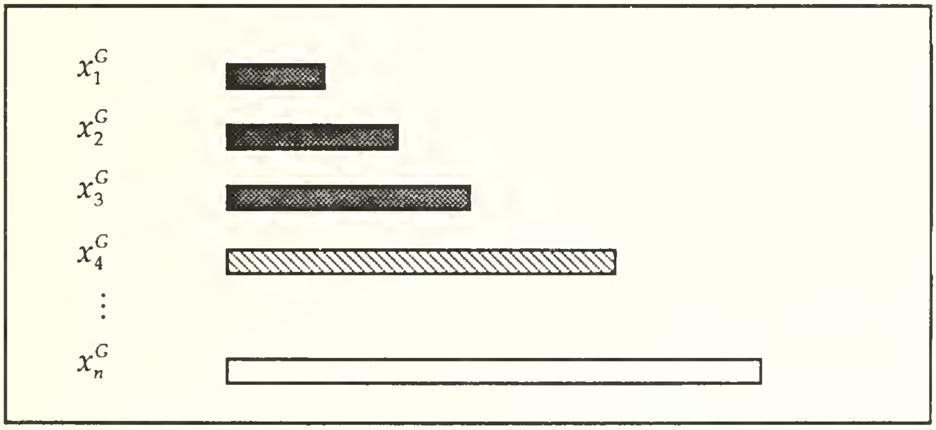


Figure 2.2a Linear Objective Function

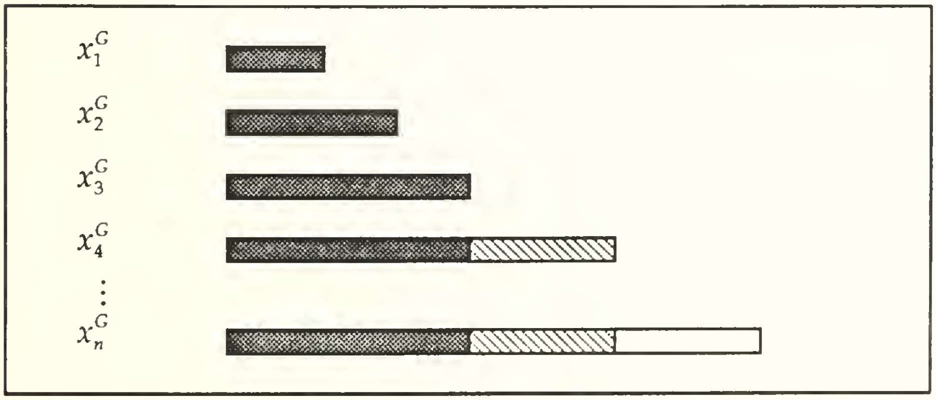


Figure 2.2b. Discounted Linear Objective Function

= accumulated cost after $|X| = 3$
= cost of adding x_4

The shaded area in Figure 2.2a indicated the magnitude of $\omega\left(X_3^G\right)$, while the shaded area of Figure 2.2b indicates the magnitude of $\omega_d\left(X_3^G\right)$. We accompany the new objective function with a modified greedy algorithm we call the greedy algorithm with discounting, shown in figure 2.3.

Denote the element $\beta_{\mathcal{M}}$ which terminates this algorithm as X^D.

Lemma 2.2. $X^D = X^G$.

Proof: We perform simple induction on the size of the set X. At initialization, $X = \phi$, and $v(x) = r(x)$ for all $x \in E$. Hence Step 1 in each algorithm is identical. Suppose that we now have generated the set $X_i^G \in \mathcal{M}$ using the standard greedy algorithm. We compare each element in the set $\{y: X_i^G \cup \{y\} \in \mathcal{M}\}$ to determine which has smallest weight, and declare the chosen element x_{i+1}^G. Because $\mathcal{M}$ has property 2.1, we know

$$\{y: X_i^G \cup \{y\} \in \mathcal{M}\} \subset \left(\{y: \phi \cup \{y\} \in \mathcal{M}\} \cap \{y: X_1^G \cup \{y\} \in \mathcal{M}\} \cap \ldots \cap \{y: X_{i-1}^G \cup [y]\}, \in \mathcal{M}\right).$$

Thus, we are guaranteed that

$$\begin{aligned} r\left(x_{i+1}^G\right) &= v\left(x_{i+1}^G\right) - \left(r\left(x_1^G\right) + r\left(x_2^G\right) + \ldots + r\left(x_i^G\right)\right) \\ &= v\left(x_{i+1}^G\right) - v\left(x_i^G\right) \end{aligned}$$

which is the minimum of the set $\left\{v(y) - v\left(x_i^G\right): X_i^G \cup \{y\} \in \mathcal{M}\right\}$. Thus, x_{i+1}^G will be selected by the greedy algorithm with discounting at stage i + 1.

Thus, we are assured that x_j^G is selected by the greedy algorithm with discounting at each stage j = 1, 2, ..., n. •

Theorem 2.3. Let X^D be the discounted linear objective function for an arbitrary, nonnegative weight function v, then

$$X^D = \arg\min_{Y \in \beta_{\mathcal{M}}} \omega_d(Y)$$

if and only if $\mathcal{M}$ is a matroid.

Proof: Since the greedy algorithm with discounting always makes the same selections as the standard greedy algorithm, this theorem follows directly from lemma 2.1. •

0. Initialize $X = \phi, r(x) = v(x) \forall x \in E, \omega_d = 0$
1. $x \leftarrow \arg\min_{y: X \cup \{y\} \in \mathcal{M}} r(y)$
2. $\omega_d \leftarrow \omega_d + [r(x)(n - |X|)]$
3. For each $y \in E - X$
   $r(y) \leftarrow r(y) - r(x)$
4. $X \leftarrow X \cup \{x\}$
5. if $X \notin \beta_{\mathcal{M}}$ then go to step 1
6. stop

Figure 2.3. The Greedy Algorithm with Discounting

Example 1. One of the more interesting weighted matroid structures is the transversal matroid. Let (S, T, A) be a bipartite graph with source node set S, destination node set T, and arc set A connecting members of S to members of T. For example, let (S T, A) be as given in Figure 2.4. Each node t in T has a specified cost v(t). The matroid minimization problem is to find the minimum weight subset of T such that each member of this subset can be matched with a unique element in S, and each element of S has a member of T matched to it. This assignment is called a transversal of the bipartite graph. Such a problem might arise where each element of T is a task requiring specialized training and each member of S is a person. An arc

exists between $p \in S$ and $t \in T$ if and only if person p is qualified to undertake the training for task t. $\{v(t): t \in T\}$ is the set of training costs. Note that training costs do not vary from person to person. The goal is to employ the members of S the greatest extent possible while minimizing the training budget. See Lawler to verify properties 2.1 and 2.2 for the transversal matroid is a matroid.

The bases for the transversal matroid are given in Table 2.1 below. All basic elements have cardinality 4. Even if there existed no complete matching from S to T, each element of $\beta_{\mathcal{M}}$ would still be of the same cardinality.

We show the execution sequence of the standard and the discounted greedy algorithms in Tables 2.2 and 2.3 below. Note that the accumulated cost and the terminating basic set are consistent with the values given in Table 2.1. Also note that r(3) decreases below 0. This only happens because 3 is not addable to {4,6} or any superset of {4,6}.

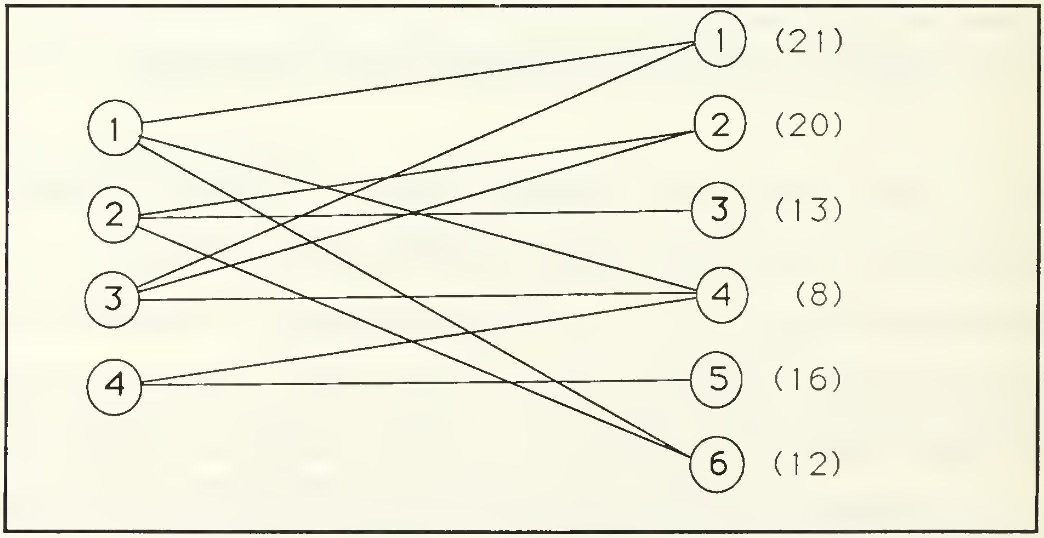


Figure 2.4. Bipartite Graph with Node Weights

TABLE 2.1. THE BASIS OF THE TRANSVERSAL MATROID

$X \in \beta_{\mathfrak{M}}$	Matching in S	Cost
{1, 2, 3, 4}	1, 3, 2, 4	62
{1, 2, 3, 5}	1, 3, 2, 4	70
{1, 2, 4, 5}	1, 2, 3, 4	65
{1, 2, 4, 6}	1, 3, 4, 2	69
{1, 2, 5, 6}	1, 3, 4, 2	77
{1, 3, 4, 5}	1, 2, 3, 4	58
{1, 4, 5, 6)	1, 3, 4, 2	57
{2, 3, 4, 5}	3, 2, 1, 4	57
{2, 4, 5, 6}	3, 1, 4, 2	56

TABLE 2.2. STANDARD GREEDY ALGORITHM EXECUTION PATH

STAGE i	X_i	ω
0	ϕ	0
1	{4}	8
2	{4,6}	20
3	{4, 5, 6}	36
4	{2, 4, 5, 6}	56

TABLE 2.3. GREEDY ALGORITHM WITH DISCOUNTING EXECUTION PATH

STAGE i	X_i	r						ω_d
		1	2	3	4	5	6	
0	ϕ	21	20	13	8	16	12	0
1	{4}	13	12	5	–	8	4	32
2	{4, 6}	9	8	1	–	4	–	44
3	{4,5,6}	5	4	–3	–	–	–	52
4	{2,4,5,6}	1	–	–7	–	–	–	56

We should note that the transversal matroid is not reducible to the graphic matroid from which the minimum weight spanning tree problem arises. Thus, the results

concerning transversal matroids with random arc weights we will present in the next section are new to the literature.

3. MATROID MINIMIZATION WITH EXPONENTIAL ELEMENT WEIGHTS

Let $\{V(y): y \in E\}$ be a set of independent, exponentially distributed random variables with arbitrary rates $\{\lambda(y): y \in E\}$, and let W_d be the associated stochastic discounted linear objective function.

We propose that a properly constructed Markov process will have the property that

i) at each transition it will make transition from $X \cup \{x\}$ with probability equal to the probability that the greedy algorithm would choose x from X;

ii) the time between entry to state X and entry into state $X \cup \{x\}$ is identically distributed with the random quantity $W(X \cup \{x\}) - W(X)$.

Once this is established, we will use the first-passage properties of the stochastic process from ϕ. to some state in $\beta_{\mathcal{M}}$ to describe the stochastic properties of the greedy algorithm and its solution. We now consider the case where our matroid is randomly weighted.

Let Z be a Markov process with statespace $\mathcal{M}$, absorbing states $\beta_{\mathcal{M}}$, and initial state ϕ. Define $\lambda(X) = \sum_{x: X \cup \{x\} \in \mathcal{M}} \lambda(x)$. Let Q be the infinitesimal generator matrix of Z, defined as

$$
\begin{aligned}
Q_{X, X \cup \{x\}} &= \frac{\lambda(x)}{n - |X|} && X, X \cup \{x\} \in \mathcal{M} \\
Q_{X, X} &= \frac{-\lambda(X)}{n - |X|} \\
Q_{X, Y} &= 0 && \text{otherwise}
\end{aligned}
$$

The fundamental result of this section is given as follows:

Theorem 3.1. For each $X \in \beta_{\mathcal{M}}$, let $P_X(t) = P[Z(t) = X]$, then

$$P\left[W_d\left(X^G\right) \le t, X^G = X\right] = P\left[Z(t) = X\right] = P_X(t).$$

Note that X^G is now a random set in $\beta_{\mathcal{M}}$

Proof. We use a sample path argument based on the current state of Z. Let $\tau_1, \tau_2, \ldots, \tau_n$ be the intertransition times of Z. At the outset, $Z(0) = \phi$.

$$W_d\left(X_1^G\right) = n\min_{y\in E} V(y) \sim n\exp\left(\sum_{y\in E}\lambda(y)\right) \sim \exp\left(\sum_{y\in E}\frac{\lambda(y)}{n}\right).$$

Once x_1^G has been chosen, we have

$$r(x) = V(x) - V\left(x_G^1\right) \sim \exp(\lambda(x))$$

for each $x \in E - \left\{x_1^G\right\}$, where the distribution is established by the strong Markov property. Thus, we have

$$W\left(X_2^G\right) - W\left(X_1^G\right) \sim \exp\left(\sum_{x:X_1^G\cup\{x\}\in\mathcal{M}}\frac{\lambda(x)}{n-1}\right).$$

Hence, we can inductively establish that

$$W\left(X_{i+1}^G\right) - W\left(X_i^G\right) \sim \exp\left(\sum_{x:X_i^G\cup\{x\}\in\mathcal{M}}\frac{\lambda(x)}{n-i}\right).$$

Our theorem follows. •

To address questions of interest regarding the performance of a randomly weighted matroid, we may use the cumulative joint distribution $\left\{P_X(t), t \ge 0, X \in \beta_{\mathcal{M}}\right\}$

to derive probabilities of interest. We now present a series of corollaries to the preceding theorem. The first result concerns the distribution of $W(X^G)$.

Modify the rate matrix Q by aggregating all the basic elements of $\mathcal{M}$ into a simple absorbing state which we will label $\beta_{\mathcal{M}}$, to created the rate matrix Q′. Let

$$
\begin{aligned}
Q'_{X,\beta_{\mathcal{M}}} &= \lambda(X) && |X| = n-1 \\
Q'_{X,Y} &= Q_{X,Y}; && |X| \neq n-1 \\
Q_{\beta_{\mathcal{M}},\beta_{\mathcal{M}}} &= 0.
\end{aligned}
$$

Let Z′ be the Markov process governed by the generator Q′.

Corollary 3.2.

$$P\left[W\left(X^G\right) \le t\right] = P\left[Z'(t) = \beta_{\mathcal{M}}\right].$$

Let $\tau_X = E\left[\text{time until absorption} \middle| Z'(0) = X\right]$. We can calculate $E\left[W\left(X^G\right)\right]$ by solving for τ_ϕ. •

We can solve for $E\left[W\left(X^G\right)\right]$ by solving for τ_ϕ according to

$$
\begin{aligned}
\tau_{\mathrm{H}} &= \frac{1}{\lambda(\mathrm{H})} && |\mathrm{H}| = \mathbf{n}-1 \\
\tau_{\mathrm{H}} &= \frac{\mathbf{n}-|\mathrm{H}|}{\lambda(\mathrm{H})} + \sum_{\mathrm{x}:\mathrm{H}\cup\{\mathrm{x}\}\in\mathcal{M}} \frac{\lambda(\mathrm{x})}{\lambda(\mathrm{H})}\tau_{\mathrm{H}\cup\{\mathrm{x}\}} && |\mathrm{H}| = \mathbf{n}-2, \mathbf{n}-3, \ldots, 1 \\
\tau_\phi &= \frac{\mathbf{n}}{\lambda(\phi)} + \sum_{\mathrm{x}:\{\mathrm{x}\}\in\mathcal{M}} \frac{\lambda(\mathrm{x})}{\lambda(\phi)}\tau_{\{\mathrm{x}\}}
\end{aligned}
$$

This is the first step analysis result found in Heymann and Sobel [1982] adapted to the process Z′. Note that $\tau_{\beta_{\mathcal{M}}} = 0$.

Often, one is not only interested in the value of $W\left(X^G\right)$. Sometimes we seek the probability that a given state is the greedy choice or the probability that a given

element is a member of X^G. The appropriate probability weight function can be easily found using the embedded Markov chain of Z′.

For each $X \in \mathcal{M}$ and x such that $X \cup \{x\} \in \mathcal{M}$ let

$$P_{X,X\cup[x]} = \lambda(x) / \lambda(X).$$

Let z be the discrete time Markov chain governed by transition probability matrix P, with $z_0 = \phi$.

Corollary 3.3. For each $X \in \beta_{\mathcal{M}}$

$$P[X^G = X] = P[z_n = X].$$

Proof: We simply take the ratio of $Q_{X,X\cup x}$ and $Q_{X,X}$. Note that the factor of $n - |X|$ cancels. •

Corollary 3.4. For $x \in E$, define p_x as

$$P[x \in X^G] = P[x \in z_n] = p_x.$$

This last result is concerned with the probability that a given element in the set E is a member of the optimal basic element. p_x is often called the *criticality index* of x, and is important because it tells us the extent to which the performance of the system depends on the random variable V(x).

There exist specific applications which require specialized performance measures such as $E[\omega(X^G)|x \in X^G]$, $P[x \in X^G|y \in X^G]$; or $P[\omega(X^G) \le t^*|x \notin X^G]$ for some t*>0. Each of these measures may be derived using standard Markov process analysis techniques. For further discussion of these techniques, see Bailey [1988].

Example 2. Returning to our task assignment example, suppose that the cost of training for each task is now exponentially distributed with mean given in figure 2.4. The appropriate Markov process was constructed with nine absorbing states, 37

transient states, and upper triangular structure. Figures 3.1 and 3.2 show the distribution and density functions, resp., for the random variable $W(X^G)$.

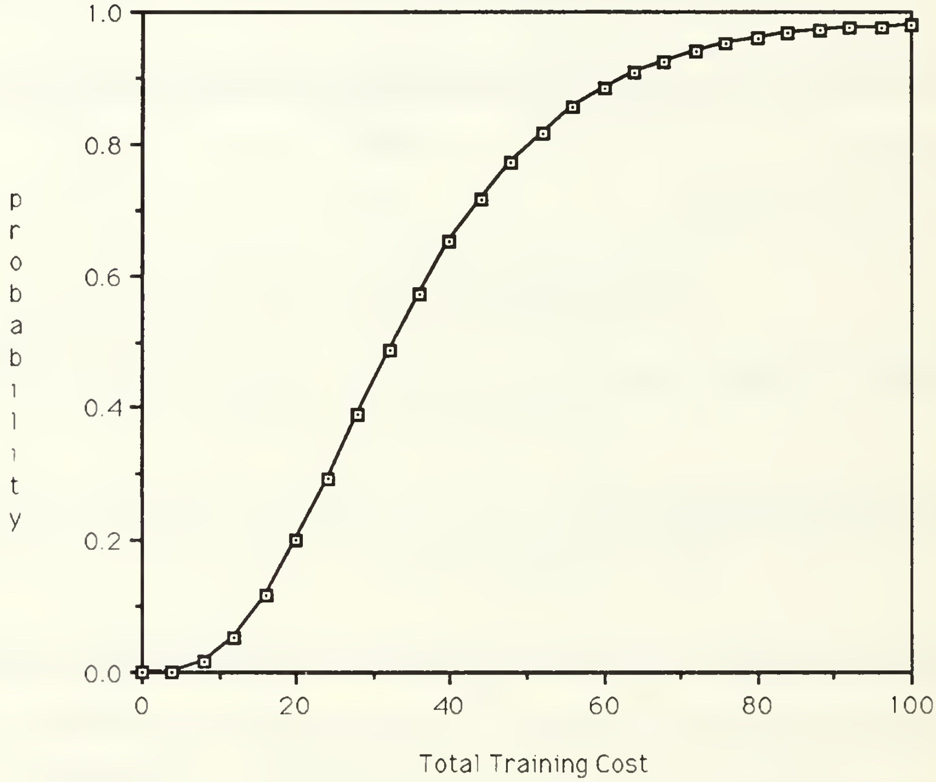


Figure 3.1 The Distribution of $W(X^G)$

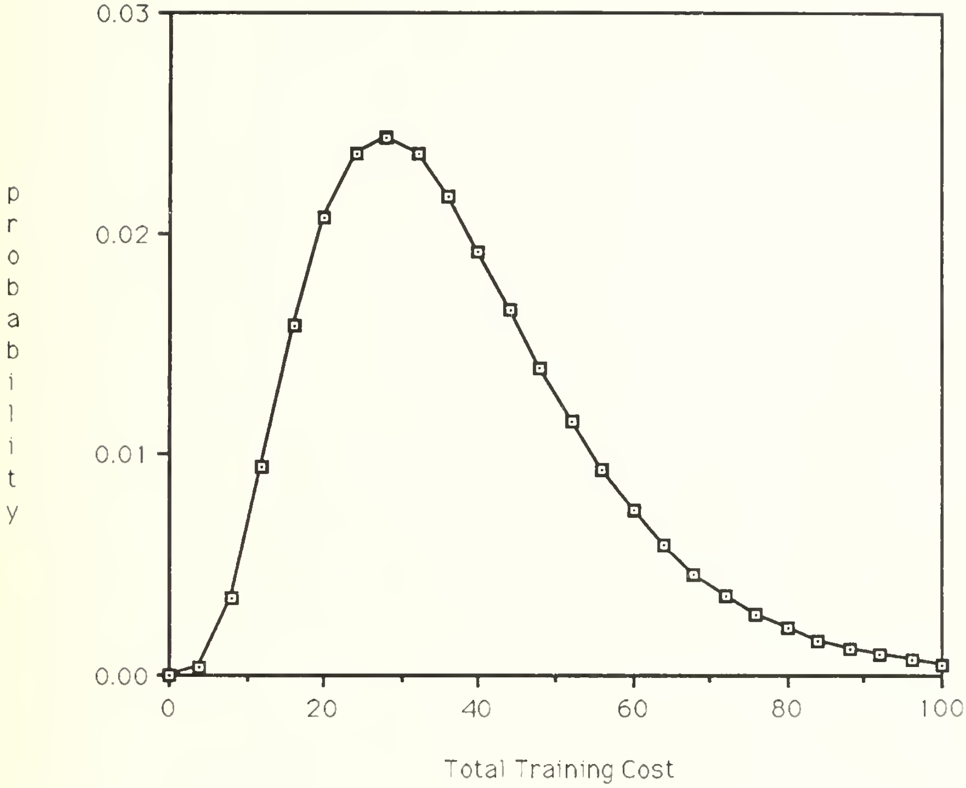


Figure 3.2. The Density of $W(X^G)$

Using the system of equations given above, we calculated the expected value of $W(X^G)$ as 35.1256. Note that this value is about 57% of the deterministic value of $W(X^G)$ given in Section 2 even though the individual expected costs in both cases are equal. This reduction in expected training costs becomes more dramatic as the size of the problem increases. In Table 3.1, we give the probability of absorption for each basic element, and we show the criticality index for each task in Figure 3.3. The criticality index in this context gives the probability that a given task is undertaken under the minimum training cost policy.

TABLE 3.1. PROBABILITY OF OPTIMALITY FOR EACH BASIC ELEMENT

$P[X^G = X]$	X
0.11124010	{1, 2, 3, 4}
0.03319019	{1, 2, 3, 5}
0.04504579	{1, 2, 4, 5}
0.13018686	{1, 2, 4, 6}
0.15166172	{1, 3, 4, 5}
0.16070031	{1, 4, 5, 6}
0.15705219	{2, 3, 4, 5}
0.19037344	{2, 4, 5, 6}

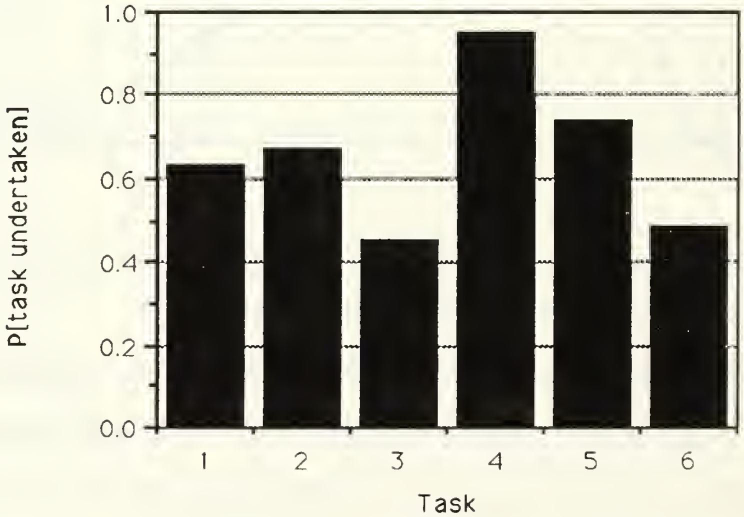


Figure 3.3. Probability each Task is Undertaken

Our analysis shows that task 4 is almost always undertaken (p_4=95%), task 5 is usually undertaken (p_5=74%), and the pairs 1 and 2, and 3 and 6 are nearly equally likely to be undertaken. The deterministic solution, {2,4,5,6}, is most likely to be the optimal solution but its probability is only 19%.

4.0 BOUNDS FOR NONEXPONENTIALLY DISTRIBUTED WEIGHTS

This section presents two methods for using exponentially weighted matroids to provide performance bounds for randomly weighted matroids. In the first case, we consider the general class of NBUE weights. We provide a simple bound on the expected weight of the optimal basic element based on a concavity argument. The second bound applies to weight distributions which have positive bounded hazard rate function on the nonnegative halfline. We develop bounds in distribution for the weight of the optimal basic element, and give formulae which quantify the worst-case tightness of the bounds. In both cases, we illustrate our methodology using the transversal matroid.

4.1 NBUE Distributions and Concave Bounds

The deterministic objective function ω_d is concave in each of its arguments. This fact allows us to state that the random variable $W_d(M)$, the weight of the minimum weight basic element when element weights are exponential, establishes a concave lower bound on $W_d(NBUE)$, where element weights have NBUE distributions with the same means as the exponentials. The power of this statement is fully revealed in the inequality

$$\omega_d \geq E[W_d(NBUE)] \geq E[W_d(M)],$$

see Stoyan [1983].

Example 3. We used our small example of the transversal matroid to compare results using two interesting families of NBUE distributions, the uniform and the Weibull families.

Our experiment involved selecting distributions ranging from those with very low coefficient of variation (0.013) to coefficients of variation near unity. We expected the performance to be close to that of the exponential system when the coefficients were high, and nearly deterministic when they were low. We felt that, even for this very small example, this transition would be interesting to observe.

For the uniform distributions we parameterized distributions using R as follows

$$f(z) = \frac{1}{2Rv(x)} I_{[v(x) - Rv(x),\, v(x) + Rv(x)]}$$

for each $x \in E$. We chose values of 0.2, 0.5, and 1.0 for R. For the Weibull distribution, given as

$$f(z) = \alpha\beta z^{\beta-1}\exp[-\alpha z^{\beta}]\, I_{[0,\,\infty]}(z),$$

we chose β = 1.5, 2, 5, and 10, and adjusted the rate parameter α so that the random weights had expected values equal to the deterministic values of v(x) given in the example in Section 2. For each distribution, we generated 1000 problem instances and generated a data set of objective function values and criticality information. We used this data to verify the above inequality. Figure 4.1 shows the relationship of the expected values of the optimal tasking cost using the distributions given above. This figure verifies the above inequality for our small example.

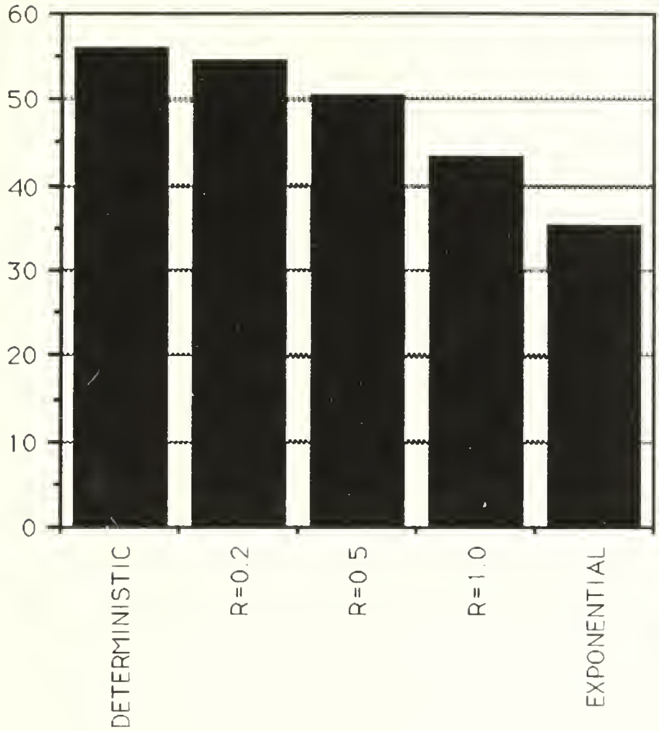


Figure 4.1a. Expected Value Bounds for Uniform Task Values

In Figure 4.2, we constructed quantile-quantile plots for the uniform and Weibull distributions. The 45 degree line represents the exponential density and the horizontal line represents the deterministic value of w=56. As we expect, these two lines nicely sandwich the distribution of W_d for each family.

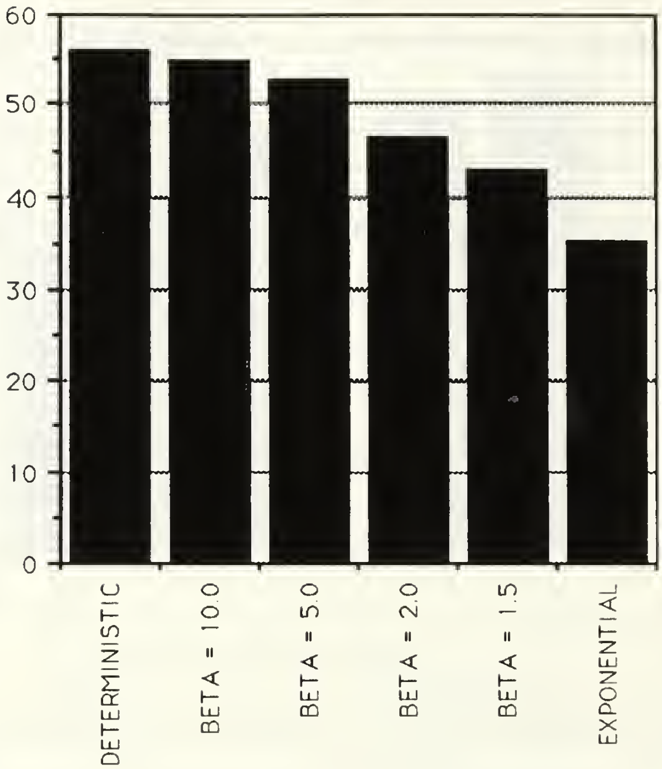


Figure 4.1b. Expected Value Bounds for Weibull Task Values

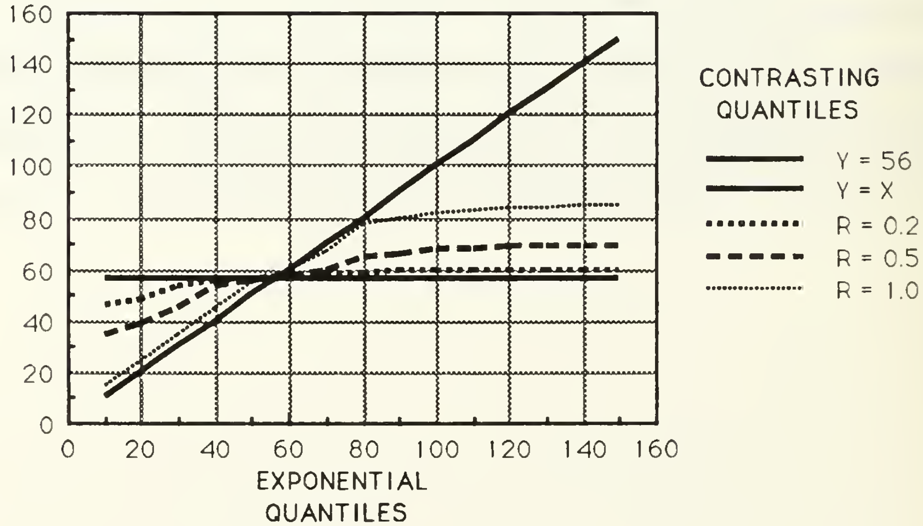


Figure 4.2a. Q-Q plot for uniform vs. exponential weights. The quantile-quantile curves progress from the 45 degree line as R decreases. The deterministic system is represented by the horizontal line at 56.

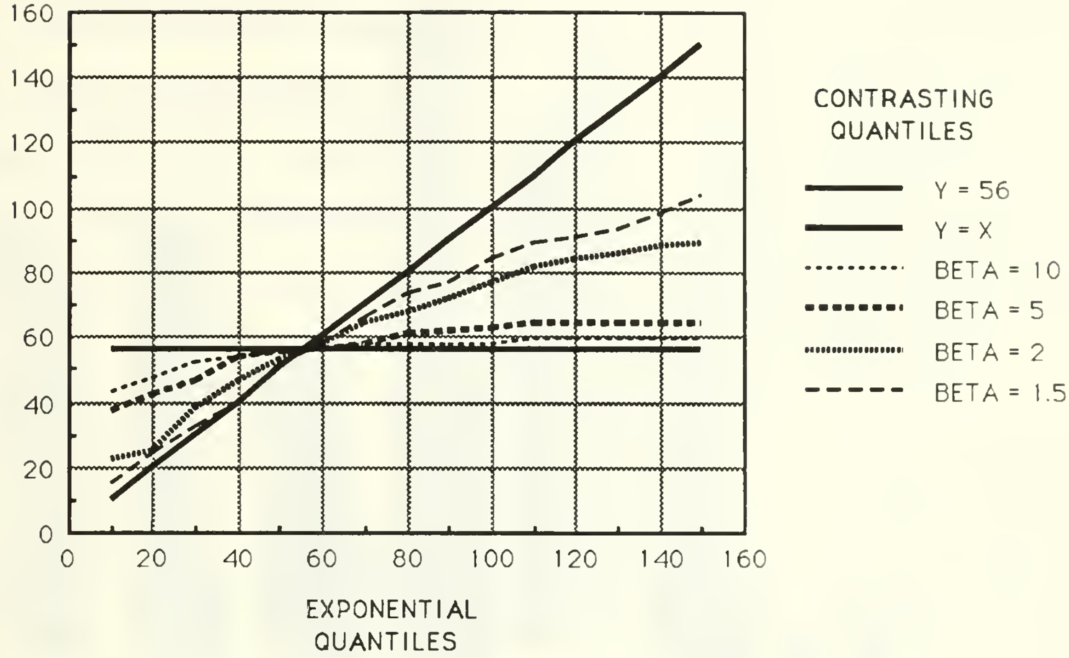


Figure 4.2b. Q-Q plot for Weibull vs. exponential weights. The curves progress from the 45 degree line as β increases. The deterministic system is represented by the horizontal line at 56.

Finally, we computed the criticality indices for the two distributional families, and present these results as bar graphs in Figure 4.3. This figure leads to some interesting insights about the bounding behavior of criticality indices. It seems that the very critical tasks, tasks four and five, have their criticality indices bounded from below by the exponential case. A similar behavior is shared by task six, though it is no more critical than tasks one or two. Tasks one through three exhibit very little systematic behavior. •

In conclusion, we have shown that the concave bounds provided by the exponential and deterministic cases hold, and that other comparisons of NBUE systems to exponential ones display interesting structures and relationships. We

more precise statements about behavior as the system grows large. Other analyses, such as that of Frieze [1985], are more applicable in large-scale systems. Finally, we should note that all of the above bounding arguments hold when each of the element weights comes from a distinct NBUE family. Thus, for any combination of IFR, IFRA, NBU and NBUE weights, we still have bounds, and the bounds probably behave in the manner we have described with respect to coefficient of variation.

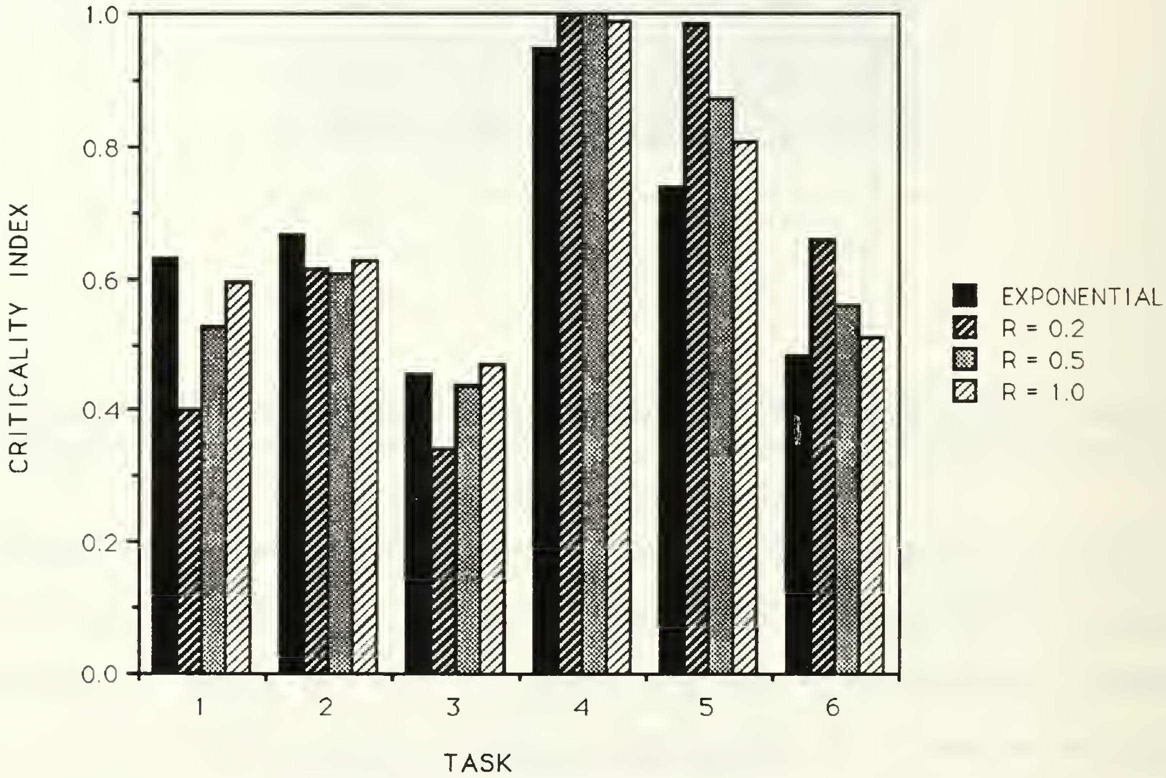


Figure 4.3a. Criticality indices from uniform task values

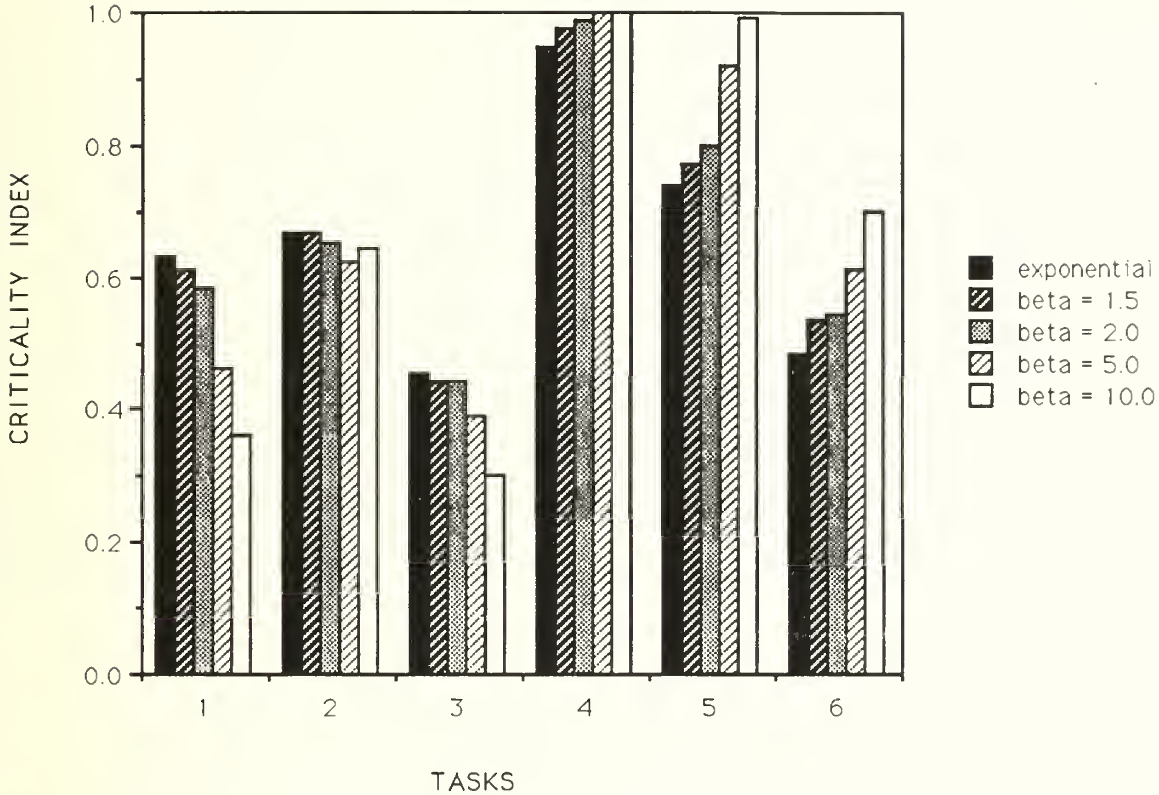


Figure 4.3b. Criticality indices from Weibull task values

4.2 Weights with Positive, Bounded Hazard Functions

In this section, we consider weights which are nearly exponential, having bounded hazard functions. Let $\{V(x) \sim F_x : x \in E\}$ be a set of weights, and define the usual hazard rate function

$$r_x(t) = \frac{f_x(t)}{(1 - F_x(t))},$$

where f_x is the distribution corresponding to the distribution function F_x. Suppose that for each $x \in E$, we have superior and inferior hazard bounds, $\lambda^s(x)$ and $\lambda^i(x)$, such that for all $t > 0$

$$t\lambda^s(x) \leq \int_0^t r_x(s)ds \leq t\lambda^i(x),$$

this model of near exponentiality being attributable to Glazebrook [1987]. Barring the case of $\lambda^s(x) = 0$, this guarantees that $V(x)$ has nontrivial support on $[0,\infty]$. Define exponential weights $V^s(x)$ and $V^i(x)$, with rates $\lambda^s(x)$ and $\lambda^i(x)$, resp., with which we can bound $V(x)$ in distribution

$$V^i(x) \leq_d V(x) \leq_d V^s(x),$$

see Barlow and Proschan [1981]. Let $W(V)$, $W(V^i)$, and $W(V^s)$ be the (random) weight of the minimum weight basis element under weights $\{V(x):x \in E\}$, $\{V^i(x),x \in E\}$, and $\{V^s(x):x \in E\}$, resp.

Theorem 4.1: $$W(V^i) \leq_d W(V) \leq_d W(V^s)$$

Proof: We will show the inequality $W(V) \leq_d W(V^s)$ holds, leaving the other inequality to the reader. Let us impose an arbitrary numbering on the elements of E, so that $E = \{e_1, e_2, ..., e_{|E|}\}$, and define random weight function V_k as $\{V^s(e_1), V^s(e_2), ..., V^s(e_k), V(e_{k+1}), V(e_{k+2}), ..., V(e_{|E|})\}$. w, the deterministic linear objective function, is monotone increasing in each argument.

We have

$$W(V) \leq_d W(V_1),$$

a result of monotonicity of w in $V(e_1)$. By a simple inductive argument, we establish that $W(V_k) \leq_d W(V_{k+1})$ for $k = 1, 2, ..., |E| - 1$. Transitivity gives us that $W(V) \leq_d W(V_{|E|}) = W(V^s)$. •

Let $x \in E$. We consider the difference $V^s(x) - V^i(x)$, where both these random variables arise from the same element of the sample space, thus they are completly dependent. Thus, if $\lambda^i(x) = \lambda^s(x) + \varepsilon(x)$, we have that

$$V^i(x) = V^s(x) \cdot \frac{\lambda^s(x)}{\lambda^s(x)+\varepsilon(x)},$$

using the method of matched realizations. Hence we calculate the probability distribution of the difference of these two weights as

$$P\left[V^s(x) - V^i(x) \geq t\right]$$

$$= P\left[V^s(x) - \frac{\lambda^s(x)}{\lambda^s(x)+\varepsilon(x)} V^s(x) \geq t\right]$$

$$= P\left[V^s(x) \geq t\left(\frac{\lambda^s(x)+\varepsilon(x)}{\varepsilon(x)}\right)\right]$$

$$= exp\left(-t\lambda^s(x)\left(\lambda^s(x)+\varepsilon(x)\right)/\varepsilon(x)\right). \tag{4.2.1}$$

We will use this result to bound $P[(W(V^s) - W(V^i) \geq t]$, using the sojourn times on the sample path to each basic element.

Let $X \in \beta$ be an (ordered) string of elements of E. Conditioned on $X = X^G$, we can use the sample path of the Markov processes $X^s(t)$ and $X^i(t)$ based on $\{V^s(x): x \in E\}$ and $\{V^i(t): x \in E\}$ with matched realizations to bound the distribution of $W(V^s) - W(V^i)$. Let T_i^s be the sojourn time of $X^s(t)$ in state X_j, then by theorem 3.1,

$$P\left[T_j^s \geq t\right] = exp\left[-t \sum_{x \in A(X_j)} \frac{\lambda^s(x)}{n-j}\right].$$

By equation (4.2.1), we have that

$$P\left[T_j^s - T_j^i \geq t \mid X=X^G\right] = exp\left(-t \sum_{x \in A(X_j)} \left[\frac{\lambda^s(x)\left(\lambda^s(x)+\varepsilon(x)\right)}{\varepsilon(x)\ (n-j)}\right]\right).$$

Thus, as $W(V^s) - W(V^i) = (T_0^s - T_0^i) + (T_1^s - T_1^i) + \ldots + (T_{n-1}^s - T_{n-1}^i)$ we get the following bound.

Lemma 4.2. Let $X \in \beta$ and $\tau_j(X) = \sum_{x \in A(X_j)} \frac{\lambda^s(x)\left(\lambda^s(x)+\varepsilon(x)\right)}{\varepsilon(x)\ (n-j)}$.

If, as we expect, $\{\tau_j(X): j=0, 1, 2, \ldots, n\text{-}1\}$ is a set of n distinct reals, then

$$P[W(V^s) - W(V^i) \geq t] = \sum_{X \in \beta} P[X=X^G]\left(\sum_{j=0}^{n-1} a_j(X)\ exp\left(-t\tau_j(X)\right)\right) \qquad (4.2.2)$$

$$\leq \max_{X \in \beta} \sum_{j=0}^{n-1} a_j(X)\ exp\left(-t\tau_j(X)\right)$$

where $a_j(X) = \prod_{\substack{k=0 \\ k \neq j}}^{n-1} \frac{\lambda(x_k)}{\left(\lambda(x_k) - \lambda(x_j)\right)}$.

Proof: Conditioned on $X = X^G$, $W(V^s) - W(V^i)$ is the convolution of n independent exponential random variables with distinct rates $\{\tau_j(X): j = 0, 1, 2, \ldots, n-1\}$. Hence $W(V^s) - W(V^i)$ is hyperexponential with the rates given by this set. The form given in (4.2.3) is found in Trevedi [1982]. •

The bound given in (4.2.2) is very sharp, and cannot be improved upon without knowledge of $P[X=X^G]$. A looser bound which does not require exact criticality data is given below. Suppose that

$$x^* = \underset{x \in E}{argmin} \frac{\lambda^s(x)(\lambda^s(x) + \varepsilon(x))}{\varepsilon(x)}.$$

Then, by 4.2.1, $P[V^s(x) - V^i(x) \geq t] \leq P[V^s(x^*) - V^i(x^*) \geq t]$ for every $x \in E$. Let

$$A_j = min_{X \in \mathcal{M},\ |X|=j}\ |A(X)|$$

Theorem 4.3:

$$P[W(V^s) - W(V^i) \geq t]$$

$$= \sum_{j=1}^{n} b_j\ exp\left(-t \left[\frac{\lambda^s(x^*)\big(\lambda^s(x^*) + \varepsilon(x^*)\big)}{\varepsilon(x^*)} \right] \frac{A_j}{(n-j)} \right)$$

where
$$b_j = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{A_k(n-j)}{A_k(n-j) - A_j(n-k)}$$

Proof: Let $X \in \beta$, then $\tau_j(x) \geq A_j \dfrac{\lambda^s(x^*)(\lambda^s(x^*) + \varepsilon(x^*))}{\varepsilon(x^*)\ (n-j)}$

$$\geq \min_{x \in \beta} \frac{\lambda^s(x^*)\ \big(\lambda^s(x^*) + \varepsilon(x^*)\big)}{\varepsilon(x^*)} \frac{A_j}{(n-j)}.$$

Thus, by rescaling the transition sojourns by $\varepsilon(x^*)/\lambda^s(x^*)\big(\lambda^s(x^*)+\varepsilon(x^*)\big)$

we derive

$$P\left[\big(W(V^s) - W(V^i)\big) \left[\frac{\varepsilon(x^*)}{\lambda^s(x^*)\big(\lambda^s(x^*) + \varepsilon(x^*)\big)} \right] \geq t \mid X = X^G \right]$$

$$\leq \prod_{j=1}^{n} b_j(X_j)\ exp\left(-t \frac{\lambda^s(x^*)\big(\lambda^s(x^*)+\varepsilon(x^*)\big)}{\varepsilon(x^*)} \frac{A_j}{(n-j)} \right)$$

where
$$b_j(X_j) = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{A_k(n-j)}{A_k(n-j)\ - A_j(n-k)}$$

and the theorem result follows by replacing x^* and X^* to bound the conditional probability for each $X \in \beta$.

Example 4. We explored the behavior of the bounds by varying the value of $\varepsilon(x)$. Using the same parameters $\{\lambda(x): x \in E\}$ as those of example 2, we varied $\lambda^s(x)$ and $l^i(x)$ such that $(1 - p)\lambda(x) = \lambda^s(x)$ and $(1 + p)\lambda(x) = \lambda^i(x)$ for $p = 0.1$, 0.2, and 0.5. The cumulative probability functions were plotted, and are presented as figure 4.4.

The results show a healthy robustness with respect to mild departures from exponentiality. The widest pair of curves, corresponding to $p = 0.5$, show the distribution for the case where $\lambda^i(x) - \lambda^s(x) = l(x)$, thus we have a 100% gap between lower and upper bounds on the hazard function. Even in this extremely weak bounding of the hazard function, we get useful results about the weight of the minimum weight basic element.

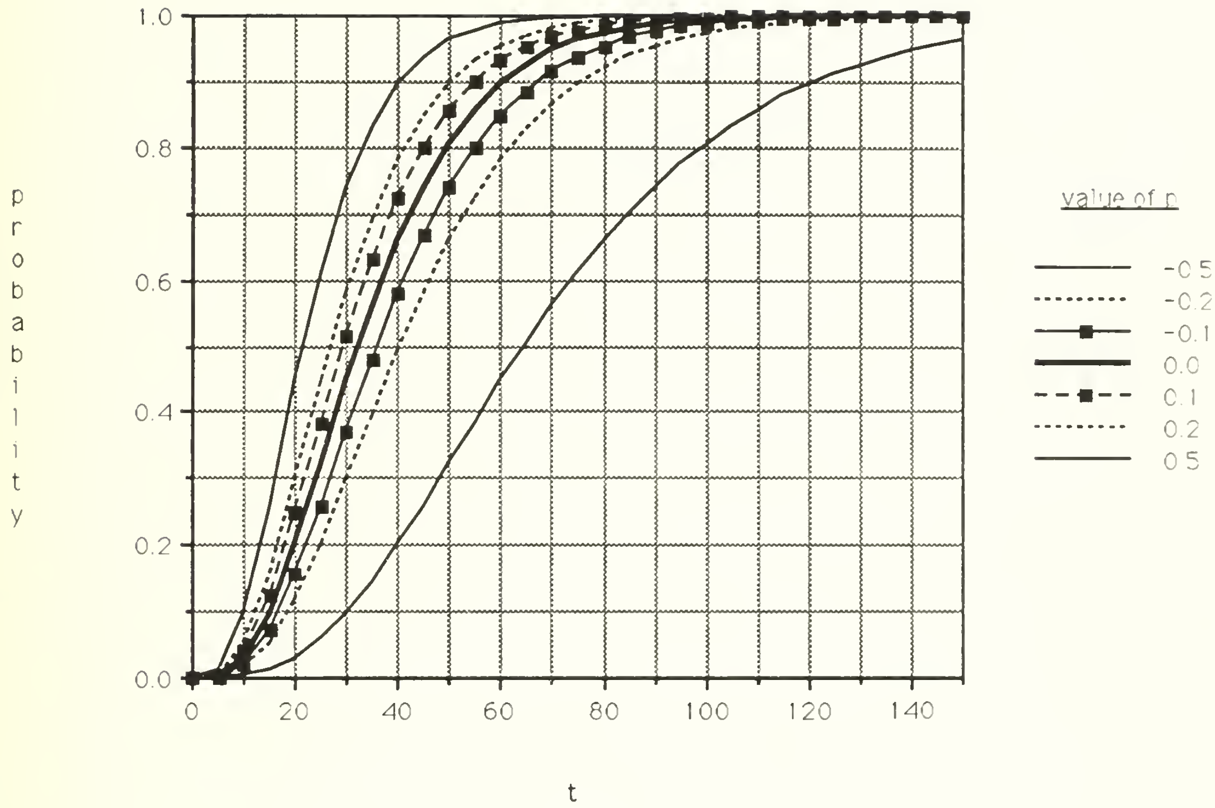
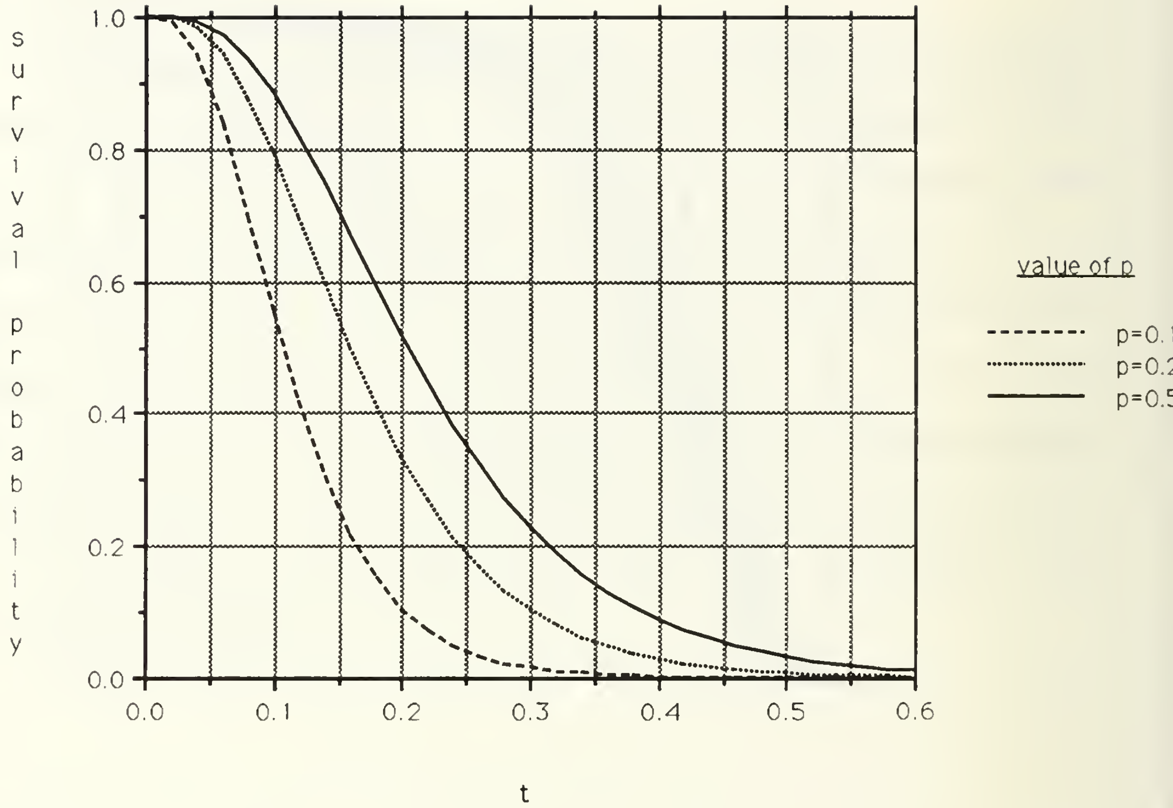


Figure 4.4. Bounds for $\lambda^s(x) = (1 - p)\lambda(x)$, $\lambda^i(x) = (1 + p)\lambda(x)$ for p = 0.1, 0.2, and 0.5. The cdf of the original problem in the bold curve marked 0.0.

We also plotted the behavior of the bound presented in theorem 4.3, the result is shown in figure 4.5. The bounding behavior is very strong for this problem, so strong that we are virtually guaranteed that the value of $W(V^s) - W(V^i)$ is less than 0.5 in all cases, even when the bounds on the hazard function r(x) are very loose. This behavior arises due to the minimization mechanism underlying W. Our intuition tells us that, as the problem structure becomes larger, the values of A_j, $j = 0, 1, \ldots, n - 1$, will become large, causing the resulting bound to be increasingly sharp.



5. CONCLUSIONS AND EXTENSIONS

We have presented a method for finding performance measures of any minimized weighted matroid optimization problem where the ground set elements have independent, exponentially distributed weights. Further, we have shown the exponential case to be useful in providing bounds on random matroids with nearly exponential weights, and NBUE weights.

Optimization problems which are amenable to the work in this paper include the semimatching problem, the minimum weight spanning tree problem, the job sequencing problem, the flow matrix synthesis problem, and the experimental

design problem. The details of each of these problems can be found in Chapter Eight of Lawler.

Each of these optimization problems must be considered well-solved for the case of exponential element weights, the solution being found in the present paper. We have further provided a general bounding scheme for any choice or mixture of NBUE distributions for ground set element weights.

Several works concerning minimum weight spanning trees present results which are asymptotic in the number of nodes in the graph examined, these results holding for generally distributed weights. The only constraint is that these distributions must be continuous from the right at $V(x)=0$. Among these papers are Frieze [1985], and Mamer and Jain [1988]. These papers make use of the fact that, as $|\{x : X \cup \{x\} \in \mathcal{M}\}| \to \infty$, we have $\min_{x: X \cup \{x\} \in \phi} V(x)$ is approximately exponentially distributed. Similar asymptotics could be performed on the general matroid problem with generally distributed element weights. Elegant results such as those given by Frieze are thus obtainable using the Markov process presented in this paper combined with some (usually complex) counting arguments.

The results presented in this paper lend further evidence supporting the usefulness of Markov processes as greedy minimization processes in combinatorial optimization problems with random weights. While methods of deterministic parametric analysis commonly used in optimization work address the case where an individual weight is allowed to vary, these methods do not address the larger problem, that the optimal objective function value is a complex function of *all* weights. The methods given in this paper are clearly more difficult to perform than the usual parametric analysis, however, we have demonstrated that the parametric methods will yield inaccurate results.

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